The local structure of modules indexed by small categories

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Write (vs/k) = the category of f.d. vector spaces over a field k.

Traditional picture

A finite (1-dim) persistence module of length n can be represented as a functor $M : \underline{n} \to (vs/k)$ where $\underline{n} = \{1 \to 2 \to 3 \to \cdots \to n\}$ is the categorical representation of the totally ordered set $\{1 < 2 < 3 < \cdots < n\}$.

Gabriel's theorem yields a decomposition of M into a direct sum of interval submodules (blocks). Each interval submodule may be further written as a direct sum of indecomposeable interval submodules of the form $[i, j]$ having dimension one at each $i \leq k \leq j$. The resulting decomposition of M is unique up to permutation of the factors.

A consequence of this is that the moduli space of isomorphism classes of 1-dim length n persistence modules is *discrete*, and moreover that any isomorphism class is completely determined by the *barcodes* computed from the above decomposition.

For more complicated underlying categories - even small 2-dim persistence modules - the corresponding moduli space of isom classes is much more topologically complex (certainly not discrete), and most modules indexed by such categories are not *tame* or even *weakly tame* (a distinction explained below).

Four questions

Let $\mathcal{C}$ denote an arbitrary small category. A $\mathcal{C}$-module is defined as a functor $M : (vs/k) \to \mathcal{C}$. This a vast generalization beyond the case $\mathcal{C} = \underline{n}$. Natural questions to ask, then are:

Q1 Do the analogues of blocks exist for general $\mathcal{C}$-modules, and are they computable?

Q2 What is the correct notion of (weakly) "tame" in this more general setting?

Q3 Is there an obstruction theory to determine when a $\mathcal{C}$-module is (weakly) tame?

Q4 When M is not (weakly) tame, is there a way to approximate it by one that is?

Results

There are roughly three main parts to the framework needed to answer these questions.

The local structure

Associated to any $\mathcal{C}$-module is a *bi-closed multi-flag* $\mathcal{F}(M)$ (defined below) referred to as its *local structure*. In most cases of interest (e.g., if $\mathcal{C}$ is any finite category and k a finite field), M has *stable* local structure. From this structure one is able to recover (in a basis-free manner) the *blocks* of M indexed on the set of admissible subcategories of $\mathcal{C}$, whose direct sum comprises the *weakly tame cover* $\mathcal{WTC}(M)$ of M, also a $\mathcal{C}$-module. This (weakly) tame cover exists regardless of whether or not M itself is tame. Each block may be expressed as a direct sum of indecomposable blocks when the nerve $N(\mathcal{C})$ is simply-connected. In the very special case $\mathcal{C}$ is the categorical representation of a finite totally ordered set, this construction recovers the interval submodule decomposition of a finite persistence module.

General position

For $\mathcal{C}$-modules with stable local structure, there exists a morphism of multi-flags

$$p : \mathcal{F}(WTC(M)) \to \mathcal{F}(M)$$

and the obstruction to this being an isomorphism is precisely measured by two types of *general position* that exist for the local structure; one associated to morphisms and one associated to objects. When the local structure $\mathcal{F}(M)$ is in general position w.r.t. the morphisms of M, the induced map p_* of associated graded objects is an isomorphism. If, in addition, $\mathcal{F}(M)$ is in general position at all of the objects of M, then p itself is an isomomorphism of multi-flags.

Inner product modules

Finally, if the $\mathcal{C}$-module M admits an *inner product*, then p is induced by a morphism of $\mathcal{C}$-modules $p' : T(M) \to M$ (i.e., $p = \mathcal{F}(p')$). In this case $T(M)$ can be viewed as the closest approximation to M by a tame $\mathcal{C}$-module. As before, when $\mathcal{F}(M)$ is in general position w.r.t. all morphisms, p' induces an isomorphism of associated graded local structures. In this case, p' induces an isomorphism of $\mathcal{C}$-modules $T(M) \xrightarrow{\cong} M \Leftrightarrow \mathcal{F}(M)$ is in general position at all objects $\Leftrightarrow$ M itself is tame. Thus, in the presence of an inner product, *the numerical general position vectors (for the set of morphisms and objects respectively) provide a complete set of discrete numerical invariants to M being tame.*

General position

A *flag* in a vector space V consists of a finite sequence of proper inclusions beginning at $\{0\}$ and ending at V:

$$\underline{W} := \{W_i\}_{0\leq i\leq n} = \{\{0\} = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_m = V\}$$

We relax this structure in two different ways. Let $Sub(V)$ denote the poset category of subspaces of V and inclusions of such.

- A *semi-flag* is a functor $F : \underline{m} \to Sub(V)$ for some m. More generally
- a *multi-flag* of V is a collection $\mathcal{F} = \{W_\alpha \subset V\}$ of subspaces of V containing $\{0\}, V$, partially ordered by inclusion, and closed under intersection. It need not be finite.

Assume now that V is equipped with an inner product. Given an element $W \subseteq V$ of a multi-flag $\mathcal{F}$ of V, let $S(W) := \{U \in \mathcal{F} \mid U \subsetneq W\}$ be the elements of $\mathcal{F}$ that are proper subsets of W, and let $SS(W) = \sum_{U\in S(W)} U$.

Then we write

$$W_{\mathcal{F}} := (SS(W) \subset W)^{\perp}, \qquad \overline{W}_{\mathcal{F}} = W/SS(W) \tag{1}$$

Definition 1. *For an IP-space V and multi-flag $\mathcal{F}$ in V, the associated graded of $\mathcal{F}$ is the set of subspaces $\mathcal{F}_* := \{W_{\mathcal{F}} \mid W \in \mathcal{F}\}$. We say that $\mathcal{F}$ is in <u>general position</u> iff V can be written as a direct sum of the elements of $\mathcal{F}_*$: $V \cong \bigoplus_{W_{\mathcal{F}}\in\mathcal{F}_*} W_{\mathcal{F}}$.*

We also define the set of subquotients

$$\overline{\mathcal{F}}_* := \{\overline{W}_{\mathcal{F}} \mid W_{\mathcal{F}} \in \mathcal{F}_*\}$$

The projection map $W_{\mathcal{F}} \to \overline{W}_{\mathcal{F}}$ is an isomorphism of vector spaces for each $W_{\mathcal{F}}$, and the association $W_{\mathcal{F}} \mapsto \overline{W}_{\mathcal{F}}$ induces an isomorphism of sets $\mathcal{F}_* \xrightarrow{\cong} \overline{\mathcal{F}}_*$. Some of the results below are most naturally formulated in terms of the associated graded of subquotients $\overline{\mathcal{F}}_*$.

[Note: $\mathcal{F}_* := \{W_\mathcal{F} \mid W \in \mathcal{F}\}$, is a set, not a multi-set. In other words we do not count multiplicities; a single element $W_\mathcal{F}$ of $\mathcal{F}_*$ may occur as the relative orthogonal complement of more than one element of the multi-flag $\mathcal{F}$.]

Proposition 1. *For any multi-flag $\mathcal{F}$ of an IP-space V, $\sum_{W_\mathcal{F} \in \mathcal{F}_*} dim(W_\mathcal{F}) \geq dim(V)$. Moreover the two are equal iff $\mathcal{F}$ is in general position.*

Definition 2. *The excess of a multi-flag $\mathcal{F}$ of an IP-space V is $e(\mathcal{F}) := \left[\sum_{W_\mathcal{F} \in \mathcal{F}_*} dim(W_\mathcal{F})\right] - dim(V)$.*

Corollary 1. *For any multi-flag $\mathcal{F}$, $e(\mathcal{F}) \geq 0$ and $e(\mathcal{F}) = 0$ iff $\mathcal{F}$ is in general position.*

Lemma 1. *If $\mathcal{G}_i$, $i = 1, 2$ are two semi-flags in the inner product space V and $\mathcal{F}$ is the smallest multi-flag containing $\mathcal{G}_1$ and $\mathcal{G}_2$ (in other words, it is the multi-flag generated by these two semi-flags), then $\mathcal{F}$ is in general position.*

On the other hand, there are simple examples of multi-flags which are not - in fact cannot be - in general position, as the following illustrates.

Example 1. *Let $\mathbb{R} \cong W_i \subset \mathbb{R}^2$ be three 1-dimensional subspaces of $\mathbb{R}^2$ intersecting in the origin, and the $\mathcal{F}$ be the multi-flag generated by this data. Then $\mathcal{F}$ is not in general position.*

[Note: This illustrates the distinction between a configuration of subspaces being of *finite type* (having finitely many isomorphism classes of configurations), and the stronger property of *tameness* (the multi-flag generated by the subspaces is in general position).]

Given an arbitrary collection of subspaces $T = \{W_\alpha\}$ of V, the multi-flag generated by T is the smallest multi-flag containing each element of T. It can be constructed as the closure of T under the operations i) inclusion of $\{0\}, V$ and ii) taking finite intersections.

If $\mathcal{F}$ is a multi-flag of V, $\mathcal{G}$ a multi-flag of W, a *morphism* of multi-flags $(L, f) : \mathcal{F} \to \mathcal{G}$ consists of

- a linear map from $L : V \to W$ and
- a map of posets $f : \mathcal{F} \to \mathcal{G}$ such that
- for each $U \in \mathcal{F}$, $L(U) \subseteq f(U)$.

$\{$*multi-flags*$\}$ will denote the category of multi-flags and morphisms of such. A morphism (L, f) of multi-flags is *closed* if for each $U \in \mathcal{F}, L(U) = f(U)$. In this case the inclusion of f is superfluous, and we will often write the morphism simply as L. L is *inverse-closed* if $L^{-1}(U') \in \mathcal{F}$ for every $U' \in \mathcal{G}$. It is *bi-closed* if it is both closed and inverse-closed.

If $L : V \to W$ is a linear map of vector spaces and $\mathcal{F}$ is a multi-flag of V, the multi-flag generated by $\{L(U) \mid U \in \mathcal{F}\} \cup \{W\}$ is a multi-flag of W which we denote by $L(\mathcal{F})$ (or $\mathcal{F}$ pushed forward by L). In the other direction, if $\mathcal{G}$ is a multi-flag of W, we write $L^{-1}[\mathcal{G}]$ for the multi-flag $\{L^{-1}[U] \mid U \in \mathcal{G}\} \cup \{\{0\}\}$ of V (i.e., $\mathcal{G}$ pulled back by L; as intersections are preserved under taking inverse images, this will be a multi-flag once we include - if needed - $\{0\}$). L defines morphisms of multi-flags $\mathcal{F} \xrightarrow{(L,\iota)} L(\mathcal{F})$, $L^{-1}[\mathcal{G}] \xrightarrow{(L,\iota')} \mathcal{G}$; however only the former need be closed, and neither need be bi-closed. The *bi-closure* of L (with respect to the multi-flags $\mathcal{F}$ and $\mathcal{G}$) is formed inductively as follows:

- Set $\mathcal{F}_0 = \mathcal{F}$, $\mathcal{G}_0 = \mathcal{G}$;
- For $n \geq 0$ let $\mathcal{F}_{n+1}$ be the multi-flag generated by $\mathcal{F}_n$ and $L^{-1}(\mathcal{G}_n)$;
- For $n \geq 1$ let $\mathcal{G}_n$ be the multi-flag generated by $\mathcal{G}_{n-1}$ and $L(\mathcal{F}_n)$;
- Let $\mathcal{F}_\infty = \varinjlim \mathcal{F}_n, \mathcal{G}_\infty = \varinjlim \mathcal{G}_n$.

Then $L : \mathcal{F}_\infty \to \mathcal{G}_\infty$ is a biclosed morphism of multi-flags. $\mathcal{F}_\infty$ and $\mathcal{G}_\infty$ are the smallest multi-flags containing $\mathcal{F}$ and $\mathcal{G}$ respectively for which the linear transformation L induces a bi-closed morphism of multi-flags $L : \mathcal{F}_\infty \to \mathcal{G}_\infty$.

Let $L : \mathcal{F} \to \mathcal{G}$ be an inverse-closed morphism of multiflags. Then the induced graded map $L_*^{-1} : \mathcal{G}_* \to \mathcal{F}_*$ is, in general, a multi-function, as the cardinality $\#\{L_*^{-1}(W_\mathcal{G})\}$ may be arbitrarily large for a given $W_\mathcal{G} \in \mathcal{G}_*$ without some restriction on the morphism or the multiflags.

Definition 3. *An inverse-closed morphism* $L : \mathcal{F} \to \mathcal{G}$ *is said to be in general position if* $\#\{L_*^{-1}(W_{\mathcal{G}})\} \leq 1$ *for all* $W \in \mathcal{G}_*$.

Alternatively, L is in general position if L_*^{-1} defines a function on the image of $L_* : \mathcal{F}_* \to \mathcal{G}_*$. Let α be the function on non-negative integers given by $\alpha(0) = \alpha(1) = 0; \alpha(n) = n - 1$ for $n \geq 2$.

Definition 4. *Let* $L : \mathcal{F} \to \mathcal{G}$ *be an inverse-closed morphism of multi-flags. The excess of* L*, denoted* $e(L)$*, is*

$$e(L) := \left[\sum_{W_{\mathcal{G}} \in \mathcal{G}_*} \alpha\left(\#\{L_*^{-1}(W_{\mathcal{G}})\}\right)\right]$$

As with multi-flags, the excess of a morphism provides a numerical invariant measuring the degree to which it fails to be in general position.

Proposition 2. *An inverse-closed morphism of multi-flags* $L : \mathcal{F} \to \mathcal{G}$ *is in general position iff* $e(L) = 0$*. Moreover, if* $L_1 : \mathcal{F} \to \mathcal{G}$*,* $L_2 : \mathcal{G} \to \mathcal{H}$ *are closed morphisms, and* L_i *is in general position for* $i = 1, 2$*, then so is* $L_2 \circ L_1$*.*

Inner products

(WIP/k) = category w/ objects inner product (IP)-spaces and morphisms linear transformations (but no compatibility is required with respect to the inner product structures on the domain and range);
(PIP/k) = wide partial subcategory of (WIP/k) whose morphisms $L : (V, <\ ,\ >_V) \to (W, <\ ,\ >_W)$ are *partial isometries* - $\widetilde{L} : ker(L)^\perp \to W$ is an isometric embedding, where $\widetilde{L}$ is the restriction of L to $ker(L)^\perp$.

A *weak inner product* on M is a factorization $M : \mathcal{C} \to (WIP/k) \xrightarrow{p_{wip}} (vs/k)$, and a $\mathcal{WIPC}$-module is a $\mathcal{C}$-module M equipped with a weak inner product. An *inner product* on M is a further factorization through a subcategory $\mathcal{D} \subset (PIP/k)$, and an $\mathcal{IPC}$-module is a $\mathcal{C}$-module M equipped with a an inner product.

A $\mathcal{C}$-module M always admits a (non-unique) weak inner product, while there are obstructions to admitting an actual inner product.

The local structure

For a $\mathcal{WIPC}$-module M a *multi-flag of* M is a functor $F : \mathcal{C} \to \{multi\text{-}flags\}$ which assigns

- to each $x \in obj(\mathcal{C})$ a multi-flag $F(x)$ of $M(x)$;
- to each $\phi_{xy} : M(x) \to M(y)$ a morphism of multi-flags $F(x) \to F(y)$

The *trivial* multi-flag F_0 of M assigns to each $x \in obj(\mathcal{C})$ the multi-flag $\{\{0\}, M(x)\}$ of $M(x)$. A multi-flag on M is *closed* resp. *inverse-closed* resp. *bi-closed* if that property holds for each morphism in the module.

A $\mathcal{WIPC}$-module M determines a multi-flag on M called the *local structure* $\mathcal{F}(M)$ of M, defined recursively at each $x \in obj(\mathcal{C})$ as follows: let $S_1(x)$ denote the set of morphisms of $\mathcal{C}$ originating at x, and $S_2(x)$ the set of morphisms terminating at x, $x \in obj(\mathcal{C})$ (note that both sets contain $Id_x : x \to x$). Then

<u>LS1</u> $\mathcal{F}_0(M)(x) =$ the multi-flag of $M(x)$ generated by

$$\{\ker(\phi_{xy} : M(x) \to M(y))\}_{\phi_{xy} \in S_1(x)} \cup \{im(\phi_{zx} : M(z) \to M(x)\}_{\phi_{zx} \in S_2(x)};$$

<u>LS2</u> For $n \geq 0$, $\mathcal{F}_{n+1}(M)(x) =$ the multi-flag of $M(x)$ generated by

LS2.1 $\phi_{xy}^{-1}[W] \subset M(x)$, where $W \in \mathcal{F}_n(M)(y)$ and $\phi_{xy} \in S_1(x)$;

LS2.2 $\phi_{zx}[W] \subset M(x)$, where $W \in \mathcal{F}_n(M)(z)$ and $\phi_{zx} \in S_2(x)$;

<u>LS3</u> $\mathcal{F}(M)(x) = \varinjlim \mathcal{F}_n(M)(x)$.

More generally, starting with a multi-flag F on M, the *local structure of* M *relative to* F is arrived at by starting in LS1 with the multi-flag generated (at each object x) by $\mathcal{F}_0(M)(x)$ and $F(x)$. The resulting direct limit is denoted $\mathcal{F}^F(M)$. The local structure of M (without superscript) is the local structure of M relative to the trivial multi-flag on M.

Proposition 3. *For any multi-flag F on M, $\mathcal{F}^F(M)$ is the smallest bi-closed multi-flag on M containing both $\mathcal{F}_0(M)$ and F.*

Definition 5. *The local structure of a $\mathcal{WIPC}$-module M is the functor $\mathcal{F}(M)$, which associates to each vertex $x \in obj(\mathcal{C})$ the multi-flag $\mathcal{F}(M)(x)$.*

Definition 6. *The local structure on M is* <u>*locally stable*</u> *at $x \in obj(\mathcal{C})$ iff there exists $N = N_x$ such that $\mathcal{F}_n(M)(x) \rightarrowtail \mathcal{F}_{n+1}(M)(x)$ is the identity map whenever $n \geq N$. It is* <u>*stable*</u> *if it is locally stable at each object. It is* <u>*strongly stable*</u> *if for all* <u>*finite*</u> *multi-flags F on M there exists $N = N(F)$ such that $\mathcal{F}^F(M)(x) = \mathcal{F}^F_N(M)(x)$ for all $x \in obj(\mathcal{C})$.*

Theorem 1. *Let M be a $\mathcal{WIPC}$-module with stable local structure. Then for all $x, y, z \in obj(\mathcal{C})$, $W \in \mathcal{F}(M)(x)$, $\phi_{zx} : M(z) \to M(x)$, and $\phi_{xy} : M(x) \to M(y)$*

1. *either $\phi_{xy}(\overline{W}_{\mathcal{F}}) = \{0\}$, or $\phi_{xy} : \overline{W}_{\mathcal{F}} \xrightarrow{\cong} \phi_{xy}(\overline{W}_{\mathcal{F}}) = \overline{\phi_{xy}(W)}_{\mathcal{F}}$, the subquotient in the associated graded $\overline{\mathcal{F}(M)}_*(y)$ induced by $\phi_{xy}(W)$;*
2. *either $im(\phi_{zx}) \cap \overline{W}_{\mathcal{F}} = \{0\}$, or for each $\overline{U}_{\mathcal{F}} \in \left(\phi_{zx}^{-1}[W]\right)_{\mathcal{F}} \subset \overline{\mathcal{F}(M)}_*(z)$, $\phi_{zx} : \overline{U}_{\mathcal{F}} \xrightarrow{\cong} \overline{W}_{\mathcal{F}}$.*

We will use the notion of general position, discussed above, to define excess.

<u>**o-excess**</u> (*excess on objects*) The *object-excess*, or *o-excess* of a $\mathcal{WIPC}$-module M is

$$e_o(M) = \sum_{x \in obj(\mathcal{C})} e(\mathcal{F}(M)(x))$$

We say $\mathcal{F}(M)$ is *in general position* at the object x iff $\mathcal{F}(M)(x)$ is in general position as defined above; in other words if $e(\mathcal{F}(M)(x)) = 0$. Then $\mathcal{F}(M)$ is *in objectwise general position* (without restriction) iff $e_o(M) = 0$; in other words, if it is in general position at all objects $x \in obj(\mathcal{C})$.

<u>**m-excess**</u> (*excess on morphisms*) The *morphism-excess*, or *m-excess* of a $\mathcal{WIPC}$-module M is

$$e_m(M) = \sum_{\phi_{xy} \in Hom(\mathcal{C})} e(\phi_{xy})$$

A morphism ϕ_{zx} is in general position iff the multi-function of associated graded sets $\phi_{zx}^{-1} : \mathcal{F}(M)_*(x) \to \mathcal{F}(M)_*(z)$ is actually a function. Globally, it follows from Proposition 2 that

Corollary 2. *For a $\mathcal{C}$-module M, $e_m(M) = 0$ iff every morphism of M is in general position.*

Note that as $M(x)$ is finite-dimensional for each $x \in obj(\mathcal{C})$, $\mathcal{F}(M)(x)$ must be locally stable at x if it is in general position (in fact, general position is a much stronger requirement).

If M is a $\mathcal{C}$-module without any additional structure, a multi-flag on M is a multi-flag on M equipped with an arbitrary $\mathcal{WIPC}$-structure. Differing choices of weak inner product on M affect the choice of relative orthogonal complements appearing in the associated graded at each object. However the constructions in LS1, LS2, and LS3 are independent of the choice of inner product, as are the definitions of excess and stability at an object and also for the module as a whole. So the results stated above for $\mathcal{WIPC}$-modules apply equally well to $\mathcal{C}$-modules.

Given a $\mathcal{WIPC}$-module M, the associated graded objects $\mathcal{F}(M)_*$ and the isomorphic $\overline{\mathcal{F}}(M)_*$ may be viewed as a $\mathcal{C}$-set; namely a functor $\mathcal{F}(M)_* \cong \overline{\mathcal{F}}(M)_* : \mathcal{C} \to \{sets\}$. This perspective is occasionally useful.

Assume now that M is equipped with an $\mathcal{IPC}$ structure. In this case all of the morphisms $\phi_{xy} : M(x) \to M(y)$ are partial isometries which map the relative orthogonal complement $W_{\mathcal{F}} = (SS(W) \subset W)^{\perp} \subset M(x)$ compatibly to $SS(\phi_{xy}(W)) \subset \phi_{xy}(W))^{\perp} \subset M(y)$ by a map which is either 0 or an isometry, by the same argument appearing in the proof of Theorem 1. A similar analysis applies for inverse images. Consequently, we can lift the above from subquotients to orthogonal complements.

Theorem 2. *Let M be an $\mathcal{IPC}$-module with stable local structure. Then for all $x, y, z \in obj(\mathcal{C})$, $W \in \mathcal{F}(M)(x)$, $\phi_{zx} : M(z) \to M(x)$, and $\phi_{xy} : M(x) \to M(y)$*

1. *either $\phi_{xy}(W_{\mathcal{F}}) = \{0\}$, or $\phi_{xy} : W_{\mathcal{F}} \xrightarrow{\cong} \phi_{xy}(W_{\mathcal{F}}) = \phi_{xy}(W)_{\mathcal{F}}$;*
2. *either $im(\phi_{zx}) \cap W_{\mathcal{F}} = \{0\}$, or for each element $U_{\mathcal{F}} \in \left(\phi_{zx}^{-1}[W]\right)_{\mathcal{F}} \subset \mathcal{F}(M)_*(z)$, $\phi_{zx} : U_{\mathcal{F}} \xrightarrow{\cong} W_{\mathcal{F}}$.*

Main results

Blocks and generalized bar codes

For a connected category $\mathcal{C}$ let $\Gamma = \Gamma(\mathcal{C})$ be its oriented graph. A subgraph $\Gamma' \subset \Gamma$ will be called *admissible* if

- it is connected;
- it is pathwise full: if $v_1e_1v_2e_2\dots v_{k-1}e_{k-1}v_k$ is an oriented path in Γ' connecting v_1 and v_k, and $(v_1 = w_1)e'_1w_2e'_2\dots w_{l-1}e'_{l-1}(w_l = v_k)$ is any other oriented path in Γ connecting v_1 and v_k then the path $v_1 = w_1e'_1w_2e'_2\dots w_{l-1}e'_{l-1}w_l$ is also in Γ'.

Any admissible subgraph Γ' of Γ determines a unique subcategory $\mathcal{C}' \subset \mathcal{C}$ for which $\Gamma(\mathcal{C}') = \Gamma'$, and we will call a subcategory $\mathcal{C}' \subset \mathcal{C}$ admissible if $\Gamma(\mathcal{C}')$ is an admissible subgraph of $\Gamma(\mathcal{C})$. If $M' \subset M$ is a sub-$\mathcal{C}$-module of the $\mathcal{C}$-module M, its *support* will refer to the full subcategory $\mathcal{C}(M') \subset \mathcal{C}$ generated by $\{x \in obj(\mathcal{C}) \mid M'(x) \neq \{0\}\}$. It is easily seen that being a submodule of M (rather than just a collection of subspaces indexed on the objects of $\mathcal{C}$) implies that the support of M', if connected, is an admissible subcatgory of $\mathcal{C}$ in the above sense. A *block* is a sub-$\mathcal{C}$-module M' of M for which $\phi_{xy} : M'(x) \xrightarrow{\cong} M'(y)$ whenever $x, y \in obj(\mathcal{C}(M'))$ (any morphism between non-zero vertex spaces of M' is an isomorphism). Finally, M' is a *generalized barcode* (GBC) for M if it is a block where $dim(M'(x)) = 1$ for all $x \in obj(\mathcal{C}(M'))$.

A $\mathcal{C}$-module M is said to be *weakly tame* iff it can be expressed as a direct sum of blocks. It is *strongly tame* or simply *tame* if, in addition, each of those blocks may be further decomposed as a direct sum of GBCs.

The (weakly) tame cover

Assume M is a $\mathcal{WIPC}$-module with stable local structure. Suppose $W \in \mathcal{F}(M)(x)$ with $0 \neq W_{\mathcal{F}} \in \mathcal{F}(M)_*(x)$. Writing $\overline{W}_{\mathcal{F}}$ as V_x we define the set $S(\overline{W}_{\mathcal{F}})_x$ of $\mathcal{WIPC}'$ modules $N : \mathcal{C}' \to \mathcal{WIP}$ where:

1. $\mathcal{C}'$ is an admissible subcategory of $\mathcal{C}$ containing the object x;
2. $N(x) = V_x$, and $N(y) \in \overline{\mathcal{F}}(M)_*(y)$ for $y \in obj(\mathcal{C}')$;
3. For each morphism $\phi_{yz} \in hom(\mathcal{C}')$, $\phi_{yz} : N(y) \xrightarrow{\cong} N(z)$;
4. N is *closed* in the following sense:
   - If $\phi_{yz} \in hom(\mathcal{C})$ with $z \in obj(\mathcal{C}')$ and $N(z) \in im(\phi_{yz} : \overline{\mathcal{F}}(M)_*(y) \to \overline{\mathcal{F}}(M)_*(z))$ then $y \in obj(\mathcal{C}')$;
   - If $\phi_{yz} \in hom(\mathcal{C}')$ with $y \in obj(\mathcal{C}')$ and $N(y) \in \overline{\mathcal{F}}(M)_*(y)$ maps isomorphically to its image in $\overline{\mathcal{F}}(M)_*(z)$ under ϕ_{yz}, then $z \in obj(\mathcal{C}')$.
5. $S(\overline{W}_{\mathcal{F}})_x$ contains all modules satisfying the above four properties.

An element $N \in S(\overline{W}_{\mathcal{F}})_x$ admits the following alternative description:

- It is a $\mathcal{C}'$-block, with $\mathcal{C}'$ as above in 1.;
- It admits a unique extension $\widetilde{N} : \mathcal{C} \to \mathcal{WIP}$ which assigns to each $y \in obj(\mathcal{C}) \backslash obj(\mathcal{C}')$ the trivial space $\{\mathbf{0}\}$
- $\mathcal{F}(N)_*(y) = \{\{\mathbf{0}\}, N(y)\}$ for each $y \in obj(\mathcal{C}')$
- $\exists$ morphisms of $\mathcal{C}$-sets $\overline{\mathcal{F}}(\widetilde{N})_* \rightarrowtail \overline{\mathcal{F}}(M)_*$, $\overline{\mathcal{F}}(M)_* \twoheadrightarrow \overline{\mathcal{F}}(\widetilde{N})_*$ whose composition is the identity, and which for each $y \in obj(\mathcal{C}')$ sends the element $N(y) \in \overline{\mathcal{F}}(\widetilde{N})_*(y)$ to $N(y) \in \overline{\mathcal{F}}(M)_*(y)$, then back to itself.

We are *not* claiming $\widetilde{N}$ is a summand of M, or even a submodule, because without extra structure such as that provided by an inner product we may not be able to compatibly lift either of these maps of associated graded $\mathcal{C}$-sets back to the modules themselves.

Define the set

$$\overline{\mathcal{SF}}(M)_* = \left(\coprod_{x \in obj\mathcal{C}} \left(\coprod_{0 \neq W_{\mathcal{F}} \in \mathcal{F}(M)_*(x)} S(\overline{W}_{\mathcal{F}})_x \right) \right) \Big/ \sim \tag{2}$$

where the equivalence relation is given by $S(\overline{W}_{\mathcal{F}})_x \ni N \sim N' \in S(\overline{W'}_{\mathcal{F}})_y$ iff there is an equality of $\mathcal{WIPC}$-module extensions

$$\widetilde{N} = \widetilde{N'} : \mathcal{C} \to \mathcal{WIP}$$

Let $\mathcal{AD}(\mathcal{C})$ denote the set of admissible subcategories of $\mathcal{C}$. For each $\mathcal{C}' \in \mathcal{AD}(\mathcal{C})$ let

$$\mathcal{WTC}(M)(\mathcal{C}') = \bigoplus_{\substack{N \in \overline{\mathcal{SF}}(M)_* \\ supp(N) = \mathcal{C}'}} \widetilde{N}$$

be the direct sum of the $\mathcal{C}$-extensions of all the elements of $\overline{\mathcal{SF}}(M)_*$ with support $\mathcal{C}'$. By construction, this is a $\mathcal{C}'$-block in the sense defined above. The *weakly tame cover of* M is

$$\mathcal{WTC}(M) = \bigoplus_{\mathcal{C}' \in \mathcal{AD}(\mathcal{C})} \mathcal{WTC}(M)(\mathcal{C}') : \mathcal{C} \to (vs/k)$$

it is a weakly tame $\mathcal{C}$-module that encodes the block structure of M regardless of whether or not M itself may be decomposed as direct sum of blocks. There is a canonical projection of associated graded local structures. For by construction

$$\overline{\mathcal{F}}(\mathcal{WTC}(M))_* = \left(\coprod_{\mathcal{C}' \in \mathcal{AD}(\mathcal{C})} \left(\coprod_{\substack{N \in \overline{\mathcal{SF}}(M)_* \\ supp(N) = \mathcal{C}'}} \overline{\mathcal{F}}(\widetilde{N})_* \backslash \{\mathbf{0}\} \right) \right) \coprod \{\mathbf{0}\}$$

yielding a projection

$$\overline{\mathcal{F}}(\mathcal{WTC}(M))_* \xrightarrow{\overline{P}(M)_* := \bigcup \iota_{\widetilde{N}}} \overline{\mathcal{F}}(M)_*$$

where for each $\widetilde{N}, \iota(\widetilde{N}) : \overline{\mathcal{F}}(\widetilde{N})_* \backslash \{\mathbf{0}\} \rightarrowtail \overline{\mathcal{F}}(\widetilde{N})_* \rightarrowtail \overline{\mathcal{F}}(M)_*$ is the natural inclusion noted above, and the union is over the indexing set $\{\mathcal{C}' \in \mathcal{AD}(\mathcal{C})\} \times \{N \in \overline{\mathcal{SF}}(M)_*, supp(N) = \mathcal{C}'\}$.

Lemma 2. *If* $e_m(M) = 0$ *(that is, if every morphism of* M *is in general position), then* $\overline{P}(M)_*$ *is an isomorphism.*

The tame cover

We now assume that M admits an $\mathcal{IPC}$-structure. As seen in Theorem 2, such structure allows us to uniformly replace subquotients with submodules, and $\overline{\mathcal{F}}(\_)$ with $\mathcal{F}(\_)$. We wish to describe the effect of this lifting on the construction of $\mathcal{WTC}(M)$.

As before let $W \in \mathcal{F}(M)(x)$ with $0 \neq W_{\mathcal{F}} \in \mathcal{F}(M)_*(x)$. Now instead let $V_x = W_{\mathcal{F}}$. We define the set $S(W_{\mathcal{F}})_x$ of V_x*-based* $\mathcal{IPC}'$ modules $N : \mathcal{C}' \to \mathcal{IP}$ where:

1. $\mathcal{C}'$ is an admissible subcategory of $\mathcal{C}$ containing the object x;

2. $N(x) = V_x$, and $N(y) \in \mathcal{F}(M)_*(y)$ for $y \in obj(\mathcal{C}')$;

3. For each morphism $\phi_{yz} \in hom(\mathcal{C}')$, $\phi_{yz} : N(y) \xrightarrow{\cong} N(z)$;

4. N is *closed* in the following sense:

   - If $\phi_{yz} \in hom(\mathcal{C})$ with $z \in obj(\mathcal{C}')$ and $N(z) \in im(\phi_{yz} : \mathcal{F}(M)_*(y) \to \mathcal{F}(M)_*(z))$ then $y \in obj(\mathcal{C}')$;
   - If $\phi_{yz} \in hom(\mathcal{C}')$ with $y \in obj(\mathcal{C}')$ and $N(y) \in \mathcal{F}(M)_*(y)$ maps isomorphically to its image in $\mathcal{F}(M)_*(z)$ under ϕ_{yz}, then $z \in obj(\mathcal{C}')$.

5. $S(W_{\mathcal{F}})_x$ contains all modules satisfying the above four properties.

Again, $N \in S(W_{\mathcal{F}})_x$ admits the following alternative description:

- It is a $\mathcal{C}'$-block, with $\mathcal{C}'$ as above in 1.;

- It admits a unique extension to an M-submodule $\widetilde{N} : \mathcal{C} \to \mathcal{IP}$ which assigns to each $y \in obj(\mathcal{C}) \backslash obj(\mathcal{C}')$ the trivial space $\{\mathbf{0}\}$

- $\mathcal{F}(N)_*(y) = \{\{\mathbf{0}\}, N(y)\}$ for each $y \in obj(\mathcal{C}')$

- There are morphisms of $\mathcal{C}$-sets $\mathcal{F}(\widetilde{N})_* \rightarrowtail \mathcal{F}(M)_*$, $\mathcal{F}(M)_* \twoheadrightarrow \mathcal{F}(\widetilde{N})_*$ whose composition is the identity, and which for each $y \in obj(\mathcal{C}')$ sends the element $N(y) \in \mathcal{F}(\widetilde{N})_*(y)$ to $N(y) \in \mathcal{F}(M)_*(y)$, then back to itself.

Define the set

$$\mathcal{SF}(M)_* = \left(\coprod_{x \in obj\mathcal{C}} \left(\coprod_{0 \neq W_{\mathcal{F}} \in \mathcal{F}(M)_*(x)} S(W_{\mathcal{F}})_x \right) \right) \Big/ \sim \tag{3}$$

where the equivalence relation is given by $S(W_{\mathcal{F}})_x \ni N \sim N' \in S(W'_{\mathcal{F}})_y$ iff there is an equality of $\mathcal{IPC}$-module extensions

$$\widetilde{N} = \widetilde{N'} : \mathcal{C} \to \mathcal{IP}$$

The weakly tame *inner product* cover of the IPC-module M is then constructed in analogy with before; one first defines the $\mathcal{IPC}'$-block

$$\mathcal{WTIPC}(M)(\mathcal{C}') = \bigoplus_{\substack{N \in \mathcal{SF}(M)_* \\ supp(N) = \mathcal{C}'}} \widetilde{N}$$

Then the *weakly tame IP-cover* of M is

$$\mathcal{WTIPC}(M) = \bigoplus_{\mathcal{C}' \in \mathcal{AD}(\mathcal{C})} \mathcal{WTIPC}(M)(\mathcal{C}') : \mathcal{C} \to (IP)$$

it is a weakly tame $\mathcal{IPC}$-module that encodes the block structure of M, just as in the case of $\mathcal{WIPC}$-modules. However, now the inner product structure allows us to construct a map of C-modules. Precisely the inclusion of $\mathcal{C}$-modules $\widetilde{N} \hookrightarrow M$ for each N above yields a $\mathcal{C}$-module surjection

$$\widetilde{P}(M) : \mathcal{WTIPC}(M) \twoheadrightarrow M$$

which then induces a surjection of multi-flags $P(M) : \mathcal{F}(\mathcal{WTIPC}(M)) \twoheadrightarrow \mathcal{F}(M)$ as well as a surjection on associated graded objects

$$P(M)_* : \mathcal{F}(\mathcal{WTIPC}(M))_* \twoheadrightarrow \mathcal{F}(M)_*$$

Lemma 3. *If $e_m(M) = 0$, $P(M)_*$ is an isomorphism.*

Lemma 4. *If $e_m(M) = e_o(M) = 0$ then $\widetilde{P}(M)$ is an isomorphim.*

Summary

Theorem 3. *Associated to any $\mathcal{WIPC}$-module M is a weakly tame $\mathcal{WIPC}$-module $\mathcal{WTC}(M)$ - the weakly tame cover of M. This cover satisfies the properties*

- *the construction is functorial in M;*
- *there is a canonical and functorial projection of associated graded sets*

$$\overline{P}(M)_* : \overline{\mathcal{F}}(\mathcal{WTC}(M))_* \twoheadrightarrow \overline{\mathcal{F}}(M)_*;$$

- *$\overline{P}(M)_*$ is an isomorphism when $e_m(M) = 0$;*
- *when $\mathcal{C}$ is h-free, the weakly tame module $\mathcal{WTC}(M)$ is tame - it decomposes as a direct sum of generalized bar codes.*

If M admits an IP-structure, then the subquotients used in the construction of $\mathcal{WPC}(M)$ may be realized as submodules of M, and the same construction applied to this family of lifts yields a weakly tame $\mathcal{IPC}$-module $\mathcal{WTIPC}(M)$ satisfying

- *the construction is functorial in M (with respect to $\mathcal{IPC}$-module morphisms);*
- *there is a canonical and functorial projection of $\mathcal{C}$-modules*

$$\widetilde{P}(M) : \mathcal{WTIPC}(M) \twoheadrightarrow M;$$

  which induces a surjection of multi-flags $P(M) : \mathcal{F}(\mathcal{WTIPC}(M)) \twoheadrightarrow \mathcal{F}(M)$ and therefore a surjection of associated graded sets

$$\overline{P}(M)_* : \mathcal{F}(\mathcal{WTIPC}(M))_* \twoheadrightarrow \mathcal{F}(M)_*$$

  which agrees with the construction of $\overline{P}(M)_*$ for $\mathcal{WIPC}$-modules;

- *$P(M)$ is an isomophism when $e_m(M) = 0$;*
- *If $e_m(M) = e_o(M) = 0$, then $\widetilde{P}(M)$ itself is an isomorphism of weakly tame $\mathcal{IPC}$-modules; moreover if M is a weakly tame $\mathcal{IPC}$-module then $e_m(M) = e_o(M) = 0$ and $\widetilde{P}(M)$ is a $\mathcal{IPC}$-module isomorphism.*
- *when $\mathcal{C}$ is h-free, the weakly tame module $\mathcal{WTIPC}(M)$ is tame - it decomposes as a direct sum of generalized bar codes.*

At least in the case of $\mathcal{IPC}$-modules, the arguments resulting in this theorem not only provide the complete obstruction to the module being tame, they also identify the obstructions themselves as a set of discrete secondary isomorphism invariants of the module. For a set S a $\mathbb{Z}_+$-valued S-vector will refer to a function $f : S \to \mathbb{Z}_+$.

Corollary 3. *Given an $\mathcal{IPC}$-module M, the*

- *$\mathbb{Z}_+$-valued $obj(\mathcal{C})$-vector $\mathbf{e}_o(M) : obj(\mathcal{C}) \to \mathbb{Z}_+$, $\mathbf{e}_o(M)(x) := e_o(M)(x)$ and the*
- *$\mathbb{Z}_+$-valued $hom(\mathcal{C})$-vector $\mathbf{e}_o(M) : hom(\mathcal{C}) \to \mathbb{Z}_+$, $\mathbf{e}_m(M)(\phi_{xy}) := e_m(M)(\phi_{xy})$*

are $\mathcal{C}$-module isomorphism invariants of M which are both identically zero iff M is weakly tame.

Some open questions

[Q1] What is the natural (correct?) notion of homotopy equivalence for $\mathcal{C}$-modules?

[Q2] For an appropriate notion of h.e., is every $\mathcal{C}$-module homotopic to one admitting an inner product structure?

[Q3] What is the precise obstruction to admitting an inner product? (second part talk will discuss this a bit)

[Q4] Is there a reasonable theory of characteristic classes for $\mathcal{C}$-modules? (some ideas here will be touched on in second talk).